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The Gacs–Kurdyumov–Levin Automaton Revisited

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We study a one-dimensional cellular automaton that was originally proposed as a candidate for exhibiting nonergodic behavior under noise. We prove that the deterministic model has the eroder property for two and only two invariant states. Moreover, we give the best possible estimates for the corresponding erosion times. We then review the results we have obtained from extensive computer simulations for the stochastic model and for a "mixed" model and argue that they suggest numerical and heuristic evidence in favour of ergodic behavior for all nonzero values of the noise parameter.

KEY WORDS: Cellular automata; ergodicity.

1. INTRODUCTION

A cellular automaton (CA) consists of a large number of individual components or cells which may take on a finite number of positions. It is updated via a local and parallel mechanism through which, at every step, the position of each cell takes on a new value depending on the configuration of cells in a finite neighborhood of it at the previous time.

Cellular automata appear under various forms as (simple) complex systems, in neural networks, in models of self-organized criticality, etc., in a variety of contexts, including biology, the social sciences, and many others. Very often one is interested in their stochastic version, the so-called probabilistic cellular automata (PCA), obtained by adding noise: that is, independently at each cell at each time, the automaton rule is followed only with some probability $1-\varepsilon$ with noise parameter $0 \leqslant \varepsilon \leqslant 1/2$.

The addition of noise is the usual way to incorporate unknown factors present in the system. For example, in computer science, the reliability of large parallel computations must be understood for noisy environments.

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The more general question is to determine, for a given CA rule and a given noise level, the behavior of the system at large times and the relation between initial state (input) and asymptotic state (output). Important in that respect is to ascertain whether the (deterministic) CA has *attracting* cell configurations, i.e., time-invariant configurations which are recovered in a finite time if one initially considers configurations that are different from them over a finite number of cells.

In that case, the CA would "wash out" finite islands of cells having arbitrary positions, immersed in a "sea" consisting of some specific cell configuration σ, to end up after some finite time with the unperturbed configuration σ which is preserved under the CA rule. This "eroder mechanism" lies at the heart of many proofs showing the existence of more than one stationary state for a low-noise PCA: these states can be thought of as perturbations of cell configurations having the eroder property for the deterministic CA rule, much as in equilibrium statistical mechanics where, under suitable conditions, the low-temperature phases can be constructed from the appropriate ground states. While such a general relation or "low-noise theory" does not yet exist for PCA, all we do know confirms this picture. The best known example is Toom's model,[1] which, in this context, was discussed in Lebowitz *et al.*[2]

In 1978, Gacs, Kurdyumov, and Levin introduced three types of one-dimensional CA having two states with an eroder property. Given the special topology of one-dimensional arrays, those examples are in fact not so easily come by as in the higher-dimensional case. This is in itself of some importance in connection with the so-called "positive rates conjecture"[3] and in the light of the discussion on nonergodicity above. Gacs *et al.*[4] wondered therefore whether the associated one-dimensional PCA had different stationary states at low noise. They were not able to conclude one way or the other, but their computer simulations led them to the conclusion that the systems were candidates for possible nonergodic behavior. In particular, their slow convergence to stationarity made them what Gacs *et al.*[4] called "quasi-nonergodic."

Nowadays it is widely believed (even by the proponents themselves, it seems) that those examples are "probably" ergodic for any finite noise, but no proof exists. On the other hand, it is still not clear whether the possibility of a nonergodic one-dimensional PCA (with some extra conditions[2]) can be realized. A counterexample to the positive rates conjecture is being constructed by Gacs.[5] An important property such a PCA should have is the ability to send messages across finite islands to let one end of the array know how good (or bad) the configuration at the other end is. This "conspiration at a distance" is necessary in one dimension (one cannot just get around the obstacle) to wash out finite islands. It is therefore useful to study this mechanism in the simplest possible case and this brings us back to the original paper.[4]

Here we revisit the simplest one-dimensional automaton having the eroder property appearing in Gacs *et al.*[4] We study this system, which is sometimes called the "soldiers model" or "unsymmetric voter model,"[6] on two levels. First we consider the deterministic rule (the CA) and give a proof of the eroder property (Section 3). We are able to derive the best possible bounds on the maximal growth of an initial island and on the time it takes for the eroder to wash out this island. We prove, moreover, that there are only two states having this property.

After we understand this mechanism we add noise the system. Extensive computer simulations allow us to predict the behavior of the relaxation time τ as a function of the size of the system L and of the noise level ε. We find that $\tau \sim e^{\alpha/\varepsilon}$ with $\alpha > 0$ a constant independent of L (for large enough systems). So we do indeed have to wait for a long time to see the system in its stationary state, but this time does not grow with the volume, no matter how low the noise level is. This is of course typically ergodic behavior and we present some further arguments for ergodicity after our discussion of the simulations (Section 4). Finally, in Section 5 we return to the broader context of the problem of phase transitions for one-dimensional PCA and we give our conclusions based on the model which is studied here.

2. THE GACS–KURDYUMOV–LEVIN MODEL

The CA we consider is a uniform chain of cells which is infinite in both directions. That is, to each site x of the one-dimensional lattice $\mathbb{Z}$, there is associated an orientation $\sigma(x) \in \{\leftarrow, \rightarrow\}$ taking on just two possible values (left or right). The full configuration $\sigma = \{\sigma(x), x \in \mathbb{Z}\}$ is updated in discrete time steps to change as a whole into $S^n(\sigma)$ at times $n = 0, 1, 2, \ldots$, with $S^0(\sigma) = \sigma$ and $S^n(\sigma) = S(S^{n-1}(\sigma))$. The operator S is the rule which defines our CA. Take any $x \in \mathbb{Z}$; then $S(\sigma)(x)$ is the outcome of the majority vote between

$$\begin{aligned} &\{\sigma(x), \sigma(x+1), \sigma(x+3)\} \qquad \text{if} \quad \sigma(x) = \rightarrow \\ &\{\sigma(x), \sigma(x-1), \sigma(x-3)\} \qquad \text{if} \quad \sigma(x) = \leftarrow \end{aligned} \tag{1}$$

Doing this simultaneously at every $x \in \mathbb{Z}$, we obtain as outcome the new configuration $S(\sigma) = \{S(\sigma)(x), x \in \mathbb{Z}\}$.

Definition 1. A *finite perturbation* η of a configuration σ is a configuration such that the set $\{x \in \mathbb{Z} : \eta(x) \neq \sigma(x)\}$ of points where they disagree is finite.

Definition 2. A configuration σ is called *attracting* if (i) $S(\sigma)=\sigma$; (ii) for any finite perturbation η there is a finite time $n(\eta)=n<\infty$ such that $S^n(\eta)=\sigma$.

If a configuration σ satisfies the first condition, we will say that it is *time-invariant*.

So far we have introduced the deterministic model. The presence of noise is modeled by associating independently to every site at every time step a finite probability of making a random choice in the updating. Let $0\leqslant\varepsilon\leqslant 1/2$, the noise level, be given. The stochastic version of the CA (which we have already refered to as the PCA) is defined as the discrete-time Markov process with transition probability

$$\prod_{x\in\mathbb{Z}} p_x(\sigma(x)\mid\eta)$$

where

$$p_x(\sigma(x)\mid\eta)=\begin{cases}1-\varepsilon & \text{if} \quad S(\eta)(x)=\sigma(x)\\ \varepsilon & \text{if} \quad S(\eta)(x)\neq\sigma(x)\end{cases} \tag{2}$$

is the probability to find the value $\sigma(x)$ at site $x\in\mathbb{Z}$ if, at the previous time, the configuration was η. Clearly, for $\varepsilon=0$ we recover the deterministic updating, while for $\varepsilon=1/2$ there is complete randomness.

It is easy to show that the regime of exponential ergodicity (where there is an exponential convergence to the unique stationary measure) extends at least to $2/5<\varepsilon\leqslant 1/2$.[7] The question to be asked therefore is whether close to the deterministic model (ε small) there is more than one invariant measure μ, i.e., different probability measures μ_k for which $\mu_k=\mu_k P$ with P the transition generator defined via (2). As we have already argued in the Introduction, it is useful in this context to examine the CA and find its (if any) attracting configurations. Their presence might indicate an interesting low-noise behavior.

3. THE DETERMINISTIC MODEL

The main result of this section is a proof of the statement[4] that the CA has two attracting states, the all-left-arrows and the all-right-arrows-state. Our proof gives the optimal estimate for the time it takes to return to these states after a finite perturbation is applied. In addition, we show that there are no attracting states other than the above-mentioned two.

Looking at the CA rule S, as defined in (1), it is clear that S commutes with reflection when the directions of the arrows are simultaneously reversed; i.e., if $R: \sigma(x) \to \sigma(-x)$ denotes reflection and $F: \sigma(x) \to -\sigma(x)$ denotes arrow-flipping, then $SRF = RFS$. It is therefore sufficient to formulate our results with respect to, say, the right arrows only. From now on we let σ^+ denote the configuration with all right arrows: $\sigma^+(x) = \to$, $\forall x \in \mathbb{Z}$.

Definition 3. We say that a configuration σ has *an island of size at most* N if for some integer n and nonnegative integer N, σ agrees with σ^+ outside of the interval $n \leqslant x \leqslant n+N-1$.

In the discussions that follow, the collection of sites included between the leftmost and the rightmost left arrows in σ will be referred to as the island of σ.

It is clear that $S(\sigma^+) = \sigma^+$, but we also have, for $N \geqslant 3$, the following result.

Theorem 1. If σ has an island size at most N, then this island never grows larger than to size at most $2N-1$.

Theorem 2. If σ has an island of size at most N, then $S^{2N-3}(\sigma) = \sigma^+$.

The cases $N = 1, 2$ can be checked directly to give $S(\sigma) = \sigma^+$, $S^2(\sigma) = \sigma^+$, respectively. Note that the estimates in Theorems 1 and 2 are optimal, as can be seen by taking

$$\sigma(x) = \leftarrow \quad \text{for } 1 \leqslant x \leqslant N \qquad \text{and} \qquad \sigma(x) = \to \quad \text{otherwise}$$

It is instructive to see (as shown in Fig. 1) how this kind of island is washed out. We will now prepare the proofs of these theorems by introducing some definitions.

Definition 4. A configuration $\sigma = \{\sigma(x), x \in \mathbb{Z}\}$ is *good to the left of* a for some $a \in \mathbb{Z}$ if it satisfies the following set of consistent conditions:

C1. $\forall x \leqslant a-2$, if $\sigma(x) = \leftarrow$, then $\sigma(x+1) = \to$.

C2. $\forall x \leqslant a-4$, if $\sigma(x) = \leftarrow$, then $\sigma(x+3) = \to$.

C3. If $\sigma(a-3) = \sigma(a) = \leftarrow$, then $\sigma(a-1) = \leftarrow$.

Definition 5. A configuration σ is *good* if σ is good to the left of a for *all* $a \in \mathbb{Z}$.

In words, to the left of site a, the configuration σ satisfying C1, C2, and C3 has neither two consecutive left arrows nor two left arrows at a

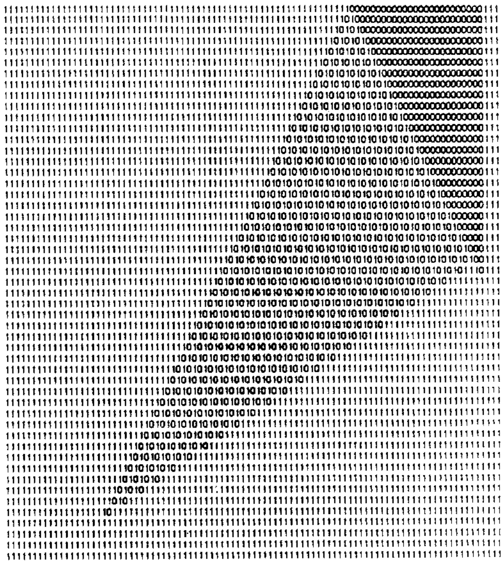

Fig. 1. A finite perturbation being washed out. The 0's represent left arrows and the 1's, right arrows.

distance 3, except near the boundary site a. For example, the configuration σ with

$$\sigma(x) = \rightarrow, \qquad \forall x \leqslant 0 \tag{3}$$

is good to the left of 2.

Lemma 1. If σ is good to the left of a, then $S(\sigma)$ is good to the left of $a+1$.

Proof. The reader can verify, by checking all possibilities, that if $S(\sigma)$ is not good to the left of $a+1$, then σ cannot be good to the left of a. For example, suppose that for $S(\sigma) = S$ we have $S(x) = S(x+1) = \leftarrow$ while $\sigma(x) = \rightarrow$ for some $x \leqslant a-1$. We must then have that $\sigma(x+1) = \leftarrow$ and therefore $\sigma(x+2) = \leftarrow$. But this is not allowed by C2 if $x \leqslant a-2$ and by C3 if $x = a-1$. ▌

Lemma 2. Let $N \geqslant 3$. If σ has an island of size at most N, then $S^{N-1}(\sigma)$ is good and $S^N(\sigma)$ has an island of size at most $2N-6$.

Proof. We may and do assume that σ agrees with σ^+ outside of the interval $1 \leqslant x \leqslant N$. From Lemma 1 and the example given in (3), $S^{N-2}(\sigma)$ is good to the left of N and $S^{N-1}(\sigma)$ is good to the left of $N+1$. But at every step n, $S^n(\sigma)(x) = \rightarrow$, $\forall x > N$, so that not only is $S^{N-1}(\sigma)$ good, but we also have that $S^{N-1}(\sigma)(N-2) = S^{N-1}(\sigma)(N-1) = \rightarrow$, and hence $S^N(\sigma)(x) = \rightarrow$, $\forall x \geqslant N-5$. At the same time, it is clear that $S^N(\sigma)(x) = \rightarrow$, $\forall x \leqslant -N$, so that the conclusion of the lemma follows. ▮

Proof of Theorem 1. At time $N-1$ the original island can have invaded at most $N-1$ other sites, growing to a size at most $2N-1$. On the other hand, since $S^{N-1}(\sigma)$ is good (Lemma 2), from this time on the island can only shrink because at each updating the rightmost left arrow moves at least two places to the left. ▮

Proof of Theorem 2. From Lemma 2 it suffices to show that $S^M(\sigma) = \sigma^+$ for every configuration σ which is good and has an island of size at most $2M$. Such a configuration σ consists of zones containing an odd number $(n \leqslant 2M-1)$ of alternating arrows with left arrows on both ends, i.e.,

$$\sigma(x-2) = \sigma(x-1) = \sigma(x) = \rightarrow, \qquad \sigma(x+1) = \leftarrow,\ \sigma(x+2) = \rightarrow, \ldots$$

alternating up to

$$\ldots,\ \sigma(x+n) = \leftarrow, \qquad \sigma(x+n+1) = \sigma(x+n+1+2) = \sigma(x+n+3) = \rightarrow$$

and these zones are separated by at least three right arrows. At the next time, $S(\sigma) = S$, and we have

$$S(x-5) = S(x-4) = S(x-3) = S(x-2) = S(x-1) = \rightarrow,$$

$$S(x) = \leftarrow,\ S(x+1) = \rightarrow, \ldots$$

alternating up to

$$\ldots,\ S(x+n-3) = \leftarrow,$$

$$S(x+n-2) = S(x+n-1) = S(x+n) = S(x+n+1) =$$

$$= S(x+n+2) = \rightarrow$$

and we get a zone of $n-2$ alternating arrows. Now all zones are separated by at least five right arrows. Therefore at time $(n-1)/2+1$ this zone has disappeared and all zones are gone at time M. ▮

It is possible to find an infinite number of cell configurations other than σ^+ and σ^- [defined of course as $\sigma^-(x) = \leftarrow$, $\forall x \in \mathbb{Z}$] which are time-invariant for the automaton rule defined by (1). Let us divide any configuration into *blocks* consisting of maximal intervals of sites containing like arrows; it is then possible to characterize the invariant configurations by the following lemma, which follows from straightforward verification:

Lemma 3. A configuration is invariant if and only if (i) every block which contains at least three right arrows extends to $+\infty$, and every block with at least three left arrows extends to $-\infty$: (ii) if two adjacent blocks each have length one, then the left one contains a right arrow.

However, only two of those invariant configurations have the eroder property, as follows from the next result.

Theorem 3. Let σ be a time-invariant configuration other than σ^+ and σ^-; then there exists $\eta \neq \sigma$, a finite perturbation of σ, which is also time-invariant.

Proof. Suppose that σ is time-invariant configuration and that it is neither σ^+ nor σ^-. If σ contains a (semi-infinite) block of at least three right arrows, then the rightmost left arrow in σ can be flipped; and similarly if σ contains a block of at least three left arrows. Otherwise, suppose that σ contains a block of length one, say consisting of a right arrow at x. By Lemma 3, the block to the left of this contains two left arrows and $\sigma(x-1)$ can be flipped. Finally, if all blocks of σ have length two, we may change

$$\cdots \leftarrow \leftarrow \rightarrow \rightarrow \leftarrow \leftarrow \rightarrow \rightarrow \leftarrow \leftarrow \rightarrow \rightarrow \cdots$$

to

$$\cdots \leftarrow \leftarrow \rightarrow \leftarrow \leftarrow \rightarrow \leftarrow \rightarrow \rightarrow \leftarrow \rightarrow \rightarrow \cdots \quad \blacksquare$$

4. THE STOCHASTIC MODEL

The stochastic model may be realized by supposing that at each individual updating there is a probability ε, where $0 < \varepsilon \leqslant 1/2$, of not following the rule, but rather of doing exactly the opposite.

The main question for this model is whether the eroder mechanism survives—in which case we will have, as previously, different "attracting" states and therefore a system that is nonergodic—or whether the eroder mechanism no longer works, so that the system is ergodic and has a unique time-invariant measure. In the latter case, this unique measure is associated with the existence of equal numbers of right and left arrows.

To get some hints of what a likely answer to this question might be, we conducted some computer simulations. A first remark that we have to make about those simulations is that, in a computer, we will obviously *not* be working with an infinite chain, but with a finite one of size L. It is clear that this means we will be trying to simulate (possible) nonergodic behavior through a system that will always be ergodic for all nonzero values of ε.

One way to get around this problem is to compute the variables we will be interested in as functions both of ε and of L, but *for many different values of* ε *and* L. Then we can hope to be able to evaluate the influence of the finiteness of our system on these variables and to predict what will happen in the limit $L \to \infty$.

We will take, as boundary conditions, a set of three fixed right arrows at either end of the finite chain. This, together with the procedure of varying L, should emulate the effect of a chain that would be infinitely extended in either direction, and filled with right arrows.

The quantities we should actually look at in our simulations were suggested by the observation[2] that on the two-dimensional space-time lattice the PCA describes an Ising-spin equilibrium system. The prototype equilibrium system showing a phase transition in two dimensions is the nearest-neighbor Ising model[8] $\{\varsigma(r)\}$ with Hamiltonian

$$\mathscr{H}(\varsigma) = -\sum_{\langle rr' \rangle} \varsigma(r)\,\varsigma(r') \tag{4}$$

Consider this last model at temperature T confined to a strip of width L on the two-dimensional lattice. An important quantity from the behavior of which we can study the phase transition is the correlation length ξ_L on this strip. It is defined via

$$\langle \varsigma(0)\,\varsigma(r) \rangle_L \sim e^{-|r|/\xi_L} \tag{5}$$

and has the meaning of the mean distance between two spins on the strip pointing in the same direction. It is well known[8] that ξ_L is a (finite) constant for all L if $T > T_c$ (the critical temperature) but that it diverges like

$$\xi_L \sim \begin{cases} L & \text{if} \quad T = T_c \\ e^{L/T} & \text{if} \quad T < T_c \end{cases} \tag{6}$$

when we consider the limit $L \to \infty$.

To study the ergodic behavior of the PCA, we propose therefore the analog of ξ_L, which we here call, in PCA language, the relaxation time τ_L. For a chain of size L, it is just the first time step at which the number of right arrows on the chain is in the interval $(L/2 - \sqrt{L}, L/2 + \sqrt{L})$, having started from the all-right-arrows state. Roughly speaking, τ_L is the number of steps needed to obtain configurations with "equal" numbers of right and left arrows.

If we take over the behavior (6), nonergodicity for small ε would be signaled by the relaxation time τ_L diverging like

$$\tau_L \sim e^{C(L)/\varepsilon} \tag{7}$$

for fixed L, $\varepsilon \downarrow 0$, and like

$$\tau_L \sim e^{C(\varepsilon)L}$$

for fixed ε, $L \uparrow \infty$. On the other hand, if the system is ergodic for small noise, we expect $C(L)$ in (7) to be bounded in L for all ε.

The computer simulations were actually carried out using a vector with L components, all of which initially set to represent right arrows, and a pseudo-random-number generator to simulate noise for each component at each updating; we then let it "run." We present below the results that we have obtained; from now on we will denote τ_L as τ for simplicity.

We took $0.05 \leqslant \varepsilon \leqslant 0.5$ and $100 \leqslant L \leqslant 6000$; for each value of (ε, L) we made an average over 100 runs to produce the points plotted in the graphs. Figure 2 shows $\ln \tau$ as a function of $1/\varepsilon$ for the different values of L considered. The data obtained from the simulations can be well fitted to straight lines, confirming the expected exponential behavior. It is interesting to remark that the lines obtained by least-squares fit are almost parallel and somewhat "huddled" together; this can already be taken as a first indication of the low sensitivity of the relaxation time to variations in the value of L.

Figure 3 shows $\ln \tau$ as a function of L for the different values of ε considered. One can immediately see that the relaxation time τ is essentially constant for $2000 \leqslant L \leqslant 6000$.

If we take the slope of the straight lines in Fig. 2 and the values of the relaxation time τ obtained for large L in Fig. 3, we can combine them to give

$$\tau \sim e^{\alpha/\varepsilon}, \qquad \alpha \sim 0.58$$

This leads us to believe, in accordance with the arguments exposed above, that this system is ergodic at least for $\varepsilon \geqslant 0.05$.

We can try to understand the reasons behind this ergodic behavior by examining what happens to an island of left arrows (in a sea of right

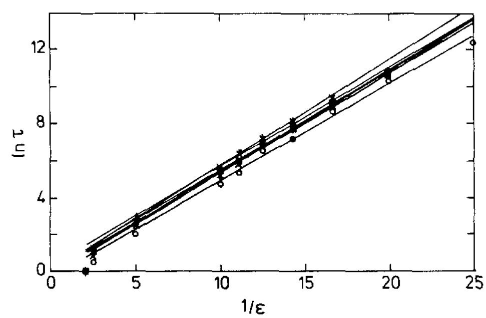

Fig. 2. Graph of $\ln \tau$ as a function of $1/\varepsilon$ for the following values of L: 100 (bottommost), 500, 800, 1000, 2000, 4000, 5000, 6000 (topmost). The points represent the data obtained via simulations and the straight lines were obtained via least-squares fit.

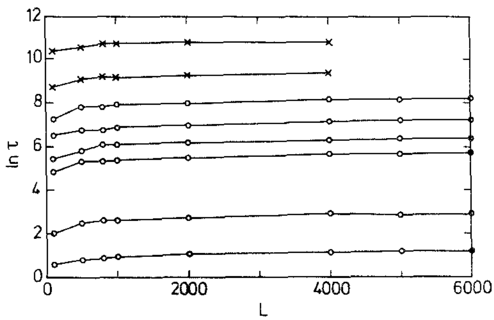

Fig. 3. Graph of $\ln \tau$ as a function of L for the following values of ε: 0.05 (topmost), 0.06, 0.07, 0.08, 0.09, 0.1, 0.2, 0.4 (bottommost). The lines simply connect the points obtained via simulation.

arrows) when noise is added to the system. We know, from the deterministic model, that two things must occur for an island to be wiped out: (i) a patch of alternating arrows is formed to the left of the perturbation, expanding in either direction; and (ii) the right end of the perturbation does not move to the right, so that the patch of alternating arrows is able to reach it and give out the "destruction" message.

When we introduce noise in the system, one or both of these conditions may be interfered with by the making of errors, so that the island may survive for a considerably longer time than in the deterministic case. To illustrate this, we present in Fig. 4 a simulation of what happens to an island under noise.

Such an island might be created as a small perturbation of the initial all-right-arrows state by a local fluctuation in the making of errors. It might then survive long enough to merge with similar perturbations, creating ever-larger regions of left arrows.

Our simulations suggest that this is the picture we see for low noise. They also suggest that this gradual buildup leads, after a long enough time, to a large excess of left arrows, which would then fill almost all the line. At this point, it is the right arrows which would constitute the perturbation; the "error-preserving" mechanism we have just described would go on, but now in the sense of a gradual buildup of the number of right arrows.

For low noise, then, we would essentially see an oscillation between large numbers of either right or left arrows, so that on average either excess would be equally probable; whereas for high noise, errors are made often enough for the system to exhibit at each step right and left arrows in roughly equal numbers and distributed evenly on the line. However, one might speculate that, in the absence of a phase transition, the system should behave similarly in the high- and low-noise cases, but with a

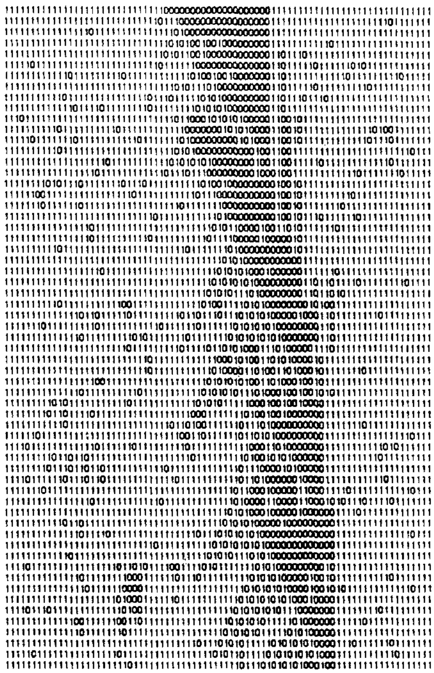

Fig. 4. The evolution of a finite perturbation for $\varepsilon = 0.05$. The 0's represent left arrows and the 1's, right arrows.

different scale; so that we would expect the oscillations above, for very large L (of the order of the relaxation time), to be actually damped.

In order to explore other properties of the stochastic model which might provide more insight into the behavior of the PCA, we conducted further simulations based on an idea suggested by Bricmont.[9] It consists of the following "mixed" system: take, as initial condition, a finite perturbation of σ^+ where the island is a block of N left arrows. Then let the island (in the sense of Definition 3) undergo stochastic evolution, while the sea of right arrows undergoes deterministic evolution. In other words, we introduce noise only *inside* the interval delimited by the outermost left arrows, and compute the time θ it takes for all left arrows to disappear.

We investigated the system with $0.00001 \leqslant \varepsilon \leqslant 0.5$ and $100 \leqslant N \leqslant 2000$. We discovered that the time θ exhibits a peculiar behavior when we consider it as a function of ε for fixed N, as can be seen in the graph presented in Fig. 5.

This behavior can be ascribed to the difference between the high- and low-noise cases alluded to above. For ε large enough, a large number of

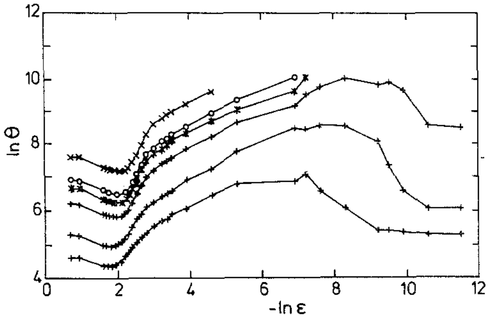

Fig. 5. Graph of ln θ as a function of $-\ln \varepsilon$ for the following values of N: 100 (bottommost), 200, 500, 800, 1000, 2000 (topmost); here $0.00001 \leqslant \varepsilon \leqslant 0.5$ and the lines simply connect the points obtained via simulation.

errors may be made early on, creating in a small number of steps configurations that are good (in the sense of Definitions 4 and 5); this may allow the island to disappear more quickly than it would if it followed a deterministic evolution, whereas for smaller values of ε, what we see at work is sensibly the same mechanism as in the PCA (where stochasticity is allowed on the whole of the chain). But since the island is finite, and subject to the "influence" of the deterministic sea of right arrows on either side of it, we finally see θ approach the deterministic value $2N-3$ for very small ε.

We believe, moreover, that this behavior may suggest that the stochastic model is indeed ergodic for any finite (i.e., nonzero) noise. Consider the value of ε for which the time θ starts decreasing (in a sense, this marks the beginning of nonergodic behavior); since this value is inversely

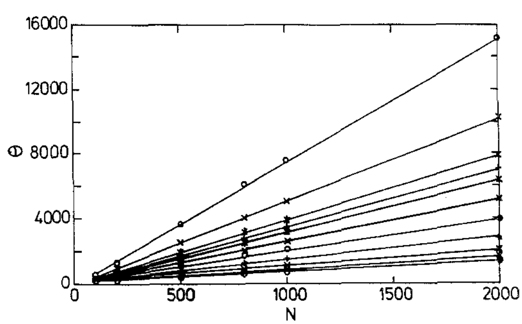

Fig. 6. Graph of θ as a function of N for the following values of ε: 0.01 (topmost), 0.02, 0.03, 0.035, 0.04, 0.05, 0.06, 0.07, 0.08, 0.09, 0.1 (bottommost). The lines were obtained by least-squares fit and the points via simulation.

dependent on N, we may imagine that as $N \uparrow \infty$, it tends to zero, so that for the real, infinite system there would be no phase transition for $\varepsilon \neq 0$.

As concerns the time for an island to disappear as a function of its size, we have that θ grows linearly with N, at least for $\varepsilon \geqslant 0.01$. We have some evidence that, for smaller ε (in the region where θ is converging to the deterministic value), we may rather have $\theta \sim N^2$, but data are as yet scant. Figure 6 shows the graph of θ as a function of N for values of ε in the interval $0.01 \leqslant \varepsilon \leqslant 0.1$.

5. ADDITIONAL REMARKS AND CONCLUSIONS

Our results indicate that the GKL model has a unique stationary measure for all finite noise levels. This conclusion is not unexpected even though the deterministic model shows certain stability against perturbing the homogeneous states.

It is indeed widely believed that stochastic perturbations of any such one-dimensional CA are ergodic. The main ingredients of the CA rule $S(\sigma)$ responsible for such behavior we take to be the following:

(i) S is one-dimensional, translation invariant, and local.

(ii) $S(\sigma^+) = \sigma^+$, $S(\sigma^-) = \sigma^-$.

(iii) If, for a given site x, $\sigma(x) = \eta(x)$ and $\sigma(y) \geqslant \eta(y)$, $\forall y \neq x$, then $S(\sigma)(x) \geqslant S(\eta)(x)$ (with the convention that "$\rightarrow$" $\geqslant$ "$\leftarrow$").

Condition (iii) implies that an arrow which has a given orientation is more likely to flip to the opposite orientation if it generally disagrees with its neighbors than if it agrees with them. For example, if $\sigma(x) = \eta(x) = \rightarrow$ and $\eta \leqslant \sigma$ in the above sense, then the corresponding arrow flip probabilities satisfy $c(x, \sigma) \leqslant c(x, \eta)$.

The main reason this "positive rates conjecture" gets such widespread support is simply that there are no known counterexamples. In addition, one sometimes refers to equilibrium statistical mechanics, where one-dimensional systems with local potentials indeed have a unique Gibbs measure (no phase transitions).

On the other hand, it has been argued that this is not the correct analog, for we know that one-dimensional PCA are in direct relation with two-dimensional equilibrium statistical mechanical systems on the space-time lattice.[2] And of course, in two dimensions, phase transitions are possible.

While we favor this two-dimensional picture, it should be pointed out that the space-time equilibrium system originates from making, with probability 2ε, independently at every point $x \in \mathbb{Z}^2$, a completely random choice $\rightarrow$ or $\leftarrow$. If we denote the corresponding random variable by $\{h_x\}_{x \in \mathbb{Z}^2}$, $h_x = \pm 1$ with equal probability, and we let $\{w_x\}_{x \in \mathbb{Z}^2}$ indicate the absence ($w_x = 1$ with probability $1 - 2\varepsilon$) or presence ($w_x = 0$ with probability 2ε) of noise at the point x, then

$$\sum_{x \in V} (1 - w_x)\, h_x \simeq 2\varepsilon \zeta_V L$$

for V a two-dimensional domain of linear size L with ζ_V approximately Gaussian (as $L \rightarrow \infty$) with mean zero and variance 1.

At the same time, the influence of the initial condition on the configuration $\{\sigma_x, x \in V\}$ should be counted as a boundary term

$$\sum_{x \in \partial V} w_x \leqslant cL$$

i.e., having a *deterministic* bound of order L, with c a constant. We assume that (i)–(iii) let the stationary system exist in some sort of ferromagnetically ordered phase in each of such domains V centered around a concentration of noise points. Then, the orientation of arrows inside V is almost everywhere the same, once a choice has been made between the orientation dictated by the sign of ζ_V and the boundary condition. Therefore, we may expect that, as in the traditional Imry–Ma argument for the random-field Ising model,[10] the configuration $\{\sigma_x\}$ will follow the fluctuation ζ_V of the neighboring random field $\{h_x\}$ and forget, as $L \rightarrow \infty$, the initial condition. This point of view lends further support to our conclusion that the GKL model is ergodic for any $\varepsilon \neq 0$.

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